

Length contraction in Very Special Relativity

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Abstract

Glashow and Cohen claim that many results of special theory of relativity (SR) like time dilation, relativistic velocity addition, etc, can be explained by using certain proper subgroups, of the Lorentz group, which collectively form the main body of Very special relativity (VSR). They did not mention about length contraction in VSR. Length contraction in VSR has not been studied at all. In this article we calculate how the length of a moving rod contracts in VSR, particularly in the *HOM*(2) version. The results are interesting in the sense that in general the length contraction formulas in VSR are different from SR but in many cases the two theories predict similar length contraction of moving rods.

1 Introduction

In the recent past Glashow and Cohen [1] proposed the interesting idea of a very special relativity. By very special relativity (VSR) the above mentioned authors meant a theory which is constituted by subgroups of the Lorentz group, but amazingly, these subgroup transformations keep the velocity of light invariant in inertial frames and time-dilation remains the same as in special relativity (SR). Velocity addition has been studied in VSR theories [2] and there has been an attempt to utilize the VSR theory as the theory of space-time transformations of dark matter candidates [3]. In a parallel development some authors have attempted to incorporate the framework of VSR in non-commutative space-times [4].

The specialty of VSR is that it can produce the constancy of light velocity and time-dilation with much smaller subgroups of the Lorentz group. Going by standard convention where \mathbf{K} specify the boost generators and \mathbf{J} specify the angular momentum generators of the full Lorentz group, there are four subgroups of the Lorentz group which exhausts all the candidates of VSR. One of the four possible versions of a theory of VSR has just two generators, namely $T_1 = K_x + J_y$ and $T_2 = K_y - J_x$. This group is called $T(2)$. If in addition to the above generators of $T(2)$ one adds another generator J_z then the resulting

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group is called $E(2)$. Instead of adding J_z if one includes K_z as the third generator in addition to the two generators of $T(2)$ one attains another subgroup of the Lorentz group which is called $HOM(2)$. Lastly, if some one includes both J_z and K_z in addition to the two generators of $T(2)$ then one obtains another subgroup of the Lorentz group which is called the $SIM(2)$. These above four subgroups of the Lorentz group which admits of local energy-momentum conservation collectively form the main body of VSR transformations.

The topic which still remains untouched is related to the topic of length contraction in VSR. Till now none of the papers on VSR has clearly stated about the way how lengths of moving rods differ from their proper length. In this article we discuss about length transformations in VSR. To do so we use the subgroup $HOM(2)$. This subgroup preserves similarity transformations or homotheties. It is seen that the length transformation formulas in VSR are dramatically different from the one we have in SR. In VSR we observe length contraction but this contraction is not equivalent to the one found in SR. More over length-contraction in VSR is direction dependent. If a rod is placed along the z -direction of the fixed frame S and the moving frame, S' , moves with an uniform velocity along the $z - z'$ axes then the length transformation formula in VSR is exactly the same as in SR. But if the rod is placed along the x -axis (y -axis) in the S frame and the S' frame moves with an uniform velocity along $x - x'$ axes ($y - y'$ axes) then the length transformation relation is not the same as found in SR. More over for very high velocities there is no length-contraction along the x or y axes motion.

The other important find in this article is that the phenomenon of length contraction is not symmetrical in the frames S and S' . By symmetrical we mean that if the rod is kept at rest in the S' frame, which is moving with respect to frame S with velocity \mathbf{u} , and the observer is in the S frame then the length contraction results does not in general match with the case where the rod is at rest in the S frame and the observer is in S' frame. This phenomenon arises because the VSR transformation which links the coordinates of the primed frame to the unprimed frame is not the same as the inverse transformation with the sign of the velocity changed. The results presented in the article can be experimentally tested in heavy ion collisions and future experiments in LHC. The experiments can conclusively state whether VSR can actually replace SR in describing the subtleties of nature.

The material in this article is presented in the following format. The next section explains the VSR transformation, particularly the $HOM(2)$ version. The notations and conventions are introduced and using them the expressions of the $HOM(2)$ transformation and its inverse transformations are deduced. Section 3 deals with the particular question of length contraction in VSR. The ultimate section 4 presents the conclusion with a brief discussion of the results derived in this article. For the sake of completeness we attach two appendices as appendix A and appendix B where the $HOM(2)$ transformation matrix and its inverse are derived explicitly.

2 Space-time transformations in the $HOM(2)$ group

The $HOM(2)$ subgroup of the Lorentz group consists of 3 generators $T_1 = K_x + J_y$, $T_2 = K_y - J_x$ and K_z where K_i 's and J_i 's are the generators of Lorentz boosts and 3-space rotations respectively. The $HOM(2)$ generators are T_1 , T_2 and K_z , satisfying the following

commutation relations [2],

$$[T_1, T_2] = 0, \quad [T_1, K_z] = iT_1, \quad [T_2, K_z] = iT_2. \quad (1)$$

In VSR if one transforms from the rest frame of a particle to a moving frame, moving with a velocity \mathbf{u} with respect to the other frame, the 4-velocity of the particle gets transformed. If the 4-velocity of the particle in the rest frame is u_0 and its 4-velocity in the moving frame be u then the 4-vectors must be like

$$u_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u = \begin{pmatrix} \gamma \\ -\gamma u_x \\ -\gamma u_y \\ -\gamma u_z \end{pmatrix}, \quad (2)$$

where $\gamma_u = 1/\sqrt{1 - \mathbf{u}^2}$ and $\mathbf{u}^2 = u_x^2 + u_y^2 + u_z^2$. The $HOM(2)$ transformation acts as:

$$L(u)u_0 = u, \quad (3)$$

where the VSR transformation matrix, $L(u)$, is given by the following equation

$$L(u) = e^{i\alpha T_1} e^{i\beta T_2} e^{i\phi K_z}. \quad (4)$$

The negative sign of the 3-vector part of the 4-vector given in Eq. (2) is chosen such that the sign matches to the corresponding 3-vector in SR. This sign convention is different from the sign convention used in Ref. [2, 1]. The appropriateness of our sign convention will be discussed once we write $L(u)$ in the matrix form. The parameters α , β and ϕ are the parameters specifying the transformation and they are given as

$$\alpha = -\frac{u_x}{1 + u_z}, \quad (5)$$

$$\beta = -\frac{u_y}{1 + u_z}, \quad (6)$$

$$\phi = -\ln(\gamma_u(1 + u_z)), \quad (7)$$

as specified in Ref. [2, 1]. The parameters specified above can be found out by using the form of the 4-vectors in Eq. (2) and the VSR transformation equation in Eq. (3). An explicit derivation of the above parameters is given in appendix A.

The form of the matrices corresponding to the three transformations $e^{i\alpha T_1}$, $e^{i\beta T_2}$ and $e^{i\phi K_z}$ can be calculated by using the standard representations of \mathbf{J} and \mathbf{K} . The following matrices encapsulate all the properties of the VSR transformations:

$$\begin{aligned} e^{i\alpha T_1} &= \begin{pmatrix} 1 + \frac{\alpha^2}{2!} & \alpha & 0 & -\frac{\alpha^2}{2!} \\ \alpha & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \\ \frac{\alpha^2}{2!} & \alpha & 0 & 1 - \frac{\alpha^2}{2!} \end{pmatrix}, \\ e^{i\beta T_2} &= \begin{pmatrix} 1 + \frac{\beta^2}{2!} & 0 & \beta & -\frac{\beta^2}{2!} \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & -\beta \\ \frac{\beta^2}{2!} & 0 & \beta & 1 - \frac{\beta^2}{2!} \end{pmatrix}, \\ e^{i\phi K_z} &= \begin{pmatrix} \cosh \phi & 0 & 0 & \sinh \phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \phi & 0 & 0 & \cosh \phi \end{pmatrix}. \end{aligned}$$

From the form of the three transformations as listed above a general $HOM(2)$ transformation, $L(u)$ as given in Eq. (4), written in terms of the velocity components u_x , u_y and u_z , can be written as,

$$L(u) = \begin{pmatrix} \gamma_u & -\frac{u_x}{1+u_z} & -\frac{u_y}{1+u_z} & -\frac{\gamma_u(u_z+\mathbf{u}^2)}{(1+u_z)} \\ -\gamma_u u_x & 1 & 0 & \gamma_u u_x \\ -\gamma_u u_y & 0 & 1 & \gamma_u u_y \\ -\gamma_u u_z & -\frac{u_x}{1+u_z} & -\frac{u_y}{1+u_z} & \frac{\gamma_u(1-\mathbf{u}^2+u_z+u_z^2)}{(1+u_z)} \end{pmatrix}. \quad (8)$$

In the above expression of $L(u)$ if we put $u_x = 0$ and $u_y = 0$ then we get a resultant $L(u_z)$ which is equivalent to the Lorentz transformation matrix in SR. The signs of the resulting SR transformation matrix matches with the sign of our $L(u_z)$. This matching of $L(u_z)$ with the corresponding SR Lorentz transformation matrix dictates the sign convention of the 3-vector of \mathbf{u} in Eq. (2). The convention of the SR Lorentz transformation followed in this article matches with the convention of Landau and Lifshitz as they explained it in Ref. [5].

In a similar way one can also calculate the inverse of the $HOM(2)$ transformation, $L^{-1}(u)$, where the inverse transformation is defined as

$$L^{-1}(u)u = u_0. \quad (9)$$

In the present case,

$$L^{-1}(u) = e^{-i\phi K_z} e^{-i\beta T_2} e^{-i\alpha T_1}. \quad (10)$$

The individual transformation matrices now are given as:

$$\begin{aligned} e^{-i\phi K_z} &= \begin{pmatrix} \cosh \phi & 0 & 0 & -\sinh \phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \phi & 0 & 0 & \cosh \phi \end{pmatrix}, \\ e^{-i\beta T_2} &= \begin{pmatrix} 1 + \frac{\beta^2}{2!} & 0 & -\beta & -\frac{\beta^2}{2!} \\ 0 & 1 & 0 & 0 \\ -\beta & 0 & 1 & \beta \\ \frac{\beta^2}{2!} & 0 & -\beta & 1 - \frac{\beta^2}{2!} \end{pmatrix}, \\ e^{-i\alpha T_1} &= \begin{pmatrix} 1 + \frac{\alpha^2}{2!} & -\alpha & 0 & -\frac{\alpha^2}{2!} \\ -\alpha & 1 & 0 & \alpha \\ 0 & 0 & 1 & 0 \\ \frac{\alpha^2}{2!} & -\alpha & 0 & 1 - \frac{\alpha^2}{2!} \end{pmatrix}, \end{aligned}$$

where the parameters α, β, ϕ are given by (5), (6), and (7). The inverse transformation matrix of the $HOM(2)$ group is given as,

$$L^{-1}(u) = \begin{pmatrix} \gamma_u & \gamma_u u_x & \gamma_u u_y & \gamma_u u_z \\ \frac{u_x}{1+u_z} & 1 & 0 & -\frac{u_x}{1+u_z} \\ \frac{u_y}{1+u_z} & 0 & 1 & -\frac{u_y}{1+u_z} \\ \frac{\gamma_u(u_z+\mathbf{u}^2)}{(1+u_z)} & \gamma_u u_x & \gamma_u u_y & \frac{\gamma_u(1-\mathbf{u}^2+u_z+u_z^2)}{(1+u_z)} \end{pmatrix}. \quad (11)$$

It can be shown that with these forms of $L(u)$ and $L^{-1}(u)$ one obtains $L(u)L^{-1}(u) = L^{-1}(u)L(u) = 1$. From the expressions in Eq. (8) and Eq. (11) it is clear that the inverse

transformation in VSR is not obtained by altering the signs of the velocity components in the $L(u)$ matrix. This property of the VSR transformations differ from the corresponding property of SR transformations. Putting $u_x = 0$ and $u_y = 0$ in the expression of $L^{-1}(u)$ we get a resultant $L(u_z)$ which is equivalent to the corresponding inverse Lorentz transformation matrix in SR.

Let us consider two inertial frames S' and S which coincide with each other at $t = t' = 0$. Suppose the S' frame is moving with velocity \mathbf{u} with respect to S frame. The coordinates of the two frames are related by

$$\mathbf{x} = L^{-1}(u) \mathbf{x}', \quad (12)$$

where $\mathbf{x} = (t, x, y, z)$ and $\mathbf{x}' = (t', x', y', z')$. Using Eq. (11) the coordinate transformation equations can be explicitly written as,

$$t = \gamma_u t' + \gamma_u u_x x' + \gamma_u u_y y' + \gamma_u u_z z' \quad (13)$$

$$x = \frac{u_x}{1 + u_z} t' + x' - \frac{u_x}{1 + u_z} z' \quad (14)$$

$$y = \frac{u_y}{1 + u_z} t' + y' - \frac{u_y}{1 + u_z} z' \quad (15)$$

$$z = \frac{\gamma_u (u_z + \mathbf{u}^2)}{(1 + u_z)} t' + \gamma_u u_x x' + \gamma_u u_y y' + \frac{\gamma_u (1 - \mathbf{u}^2 + u_z + u_z^2)}{(1 + u_z)} z' \quad (16)$$

From the above equations one can explicitly verify that $ds^2 = ds'^2$, where the invariant line-element squared is $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. As the square of the line-element remains invariant under $HOM(2)$ transformations one can derive a time dilation formula in this case. The result matches exactly with that of SR.

3 Length of a moving rod in VSR

In this section we will discuss the length contraction formulas in VSR. As VSR does have a preferred direction, which is along the z -axis of the S frame, one cannot arbitrarily rotate the coordinate systems to suit ones need as in SR. In this case one has to rely more on mathematical description of the physical problem and the concept of isotropy has to be kept aside. The most general treatment of the length contraction formula requires the moving rod to be arbitrarily placed in the moving frame which can have any arbitrary velocity (although the magnitude of velocity must be smaller than one). The general setting of the length contraction problem is too complicated and cumbersome in VSR as because the transformation equations are themselves complicated as compared to SR. But a meaningful approach and some interesting results can be obtained by some specific cases and in this section we will try to elucidate these points explicitly.

3.1 The rod is at rest in the S frame

In this section we discuss the issue about length transformations in VSR. We will focus our attention particularly to the $HOM(2)$ transformations. For the first case we suppose that a rod is at rest along the x -axis in the S frame. The length of the rod is $\Delta x = x_2 - x_1 \equiv l_0$. An observer in the S' frame, which is moving with a velocity \mathbf{u} with respect to the S frame, can measure the length of the rod in his/her frame. For the measurement of the length of

the rod in motion one has to know the coordinates of the two ends of the rod $((x'_1, y'_1, z'_1)$ and (x'_2, y'_2, z'_2)) simultaneously (at t'). From the form of the coordinate transformations given in the last section we can write that

$$\begin{aligned} x_1 &= \frac{u_x}{1+u_z}t' + x'_1 - \frac{u_x}{1+u_z}z'_1, \\ y_1 &= \frac{u_y}{1+u_z}t' + y'_1 - \frac{u_y}{1+u_z}z'_1, \\ z_1 &= \frac{\gamma_u(u_z + \mathbf{u}^2)}{(1+u_z)}t' + \gamma_u u_x x'_1 + \gamma_u u_y y'_1 + \frac{\gamma_u(1 - \mathbf{u}^2 + u_z + u_z^2)}{(1+u_z)}z'_1, \end{aligned}$$

which connects the coordinates of one end of the rod in S frame to its corresponding coordinates in the S' frame. A similar set of relations can be written for x_2 , y_1 and z_1 (coordinates of the other end of the rod in S frame) as

$$\begin{aligned} x_2 &= \frac{u_x}{1+u_z}t' + x'_2 - \frac{u_x}{1+u_z}z'_2, \\ y_1 &= \frac{u_y}{1+u_z}t' + y'_2 - \frac{u_y}{1+u_z}z'_2, \\ z_1 &= \frac{\gamma_u(u_z + \mathbf{u}^2)}{(1+u_z)}t' + \gamma_u u_x x'_2 + \gamma_u u_y y'_2 + \frac{\gamma_u(1 - \mathbf{u}^2 + u_z + u_z^2)}{(1+u_z)}z'_2. \end{aligned}$$

It is interesting to note that although y and z coordinates remain the same for the two ends of the rod in the S frame, in the S' frame it does not remain so. Subtracting the first triplet of equations from the second triplet we have

$$l_0 = \Delta x' - \frac{u_x}{1+u_z}\Delta z', \quad (17)$$

$$0 = \Delta y' - \frac{u_y}{1+u_z}\Delta z', \quad (18)$$

$$0 = \gamma_u u_x \Delta x' + \gamma_u u_y \Delta y' + \frac{\gamma_u(1 - \mathbf{u}^2 + u_z + u_z^2)}{(1+u_z)}\Delta z', \quad (19)$$

where $\Delta x' \equiv x'_2 - x'_1$, $\Delta y' \equiv y'_2 - y'_1$ and $\Delta z' \equiv z'_2 - z'_1$. Using the first two equations, in the above set of equations, one can deduce from Eq. (19) that

$$\Delta z' = -u_x l_0. \quad (20)$$

Using Eq. (17), Eq. (18) and the above equation one can show that

$$l = l_0 \sqrt{\left(1 - \frac{u_x^2}{1+u_z}\right)^2 + \frac{u_x^2 u_y^2}{(1+u_z)^2} + u_x^2}, \quad (21)$$

where

$$l \equiv \sqrt{(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2}, \quad (22)$$

is the length of the rod in the S' frame. It may happen that the rod, at rest, is placed along the x axis of the S frame while some of the velocity components of \mathbf{u} are zero. In this case Eq. (21) gets simplified. If $u_y \neq 0$ and $u_z = u_x = 0$, or $u_z \neq 0$ and $u_x = u_y = 0$,

in both cases the length of the rod remains the same. On the other hand if $u_x \neq 0$ and $u_y = u_z = 0$ we will have length contraction and Eq. (21) becomes

$$l = l_0 \sqrt{(1 - u_x^2)^2 + u_x^2}, \quad (23)$$

which shows that in general there will be a length contraction but the amount of contraction depends on u_x . For $u_x \rightarrow 1$ length contraction disappears.

If the rod at rest in the S frame was kept along the y axis with coordinates (x_1, y_1, z_1) and (x_1, y_2, z_1) then analyzing in a similar way one can write

$$l = l_0 \sqrt{\left(1 - \frac{u_y^2}{1 + u_z}\right)^2 + \frac{u_y^2 u_x^2}{(1 + u_z)^2} + u_y^2}, \quad (24)$$

where now $l_0 = y_2 - y_1$ and l is as given in Eq. (22). If $u_x \neq 0$ and $u_y = u_z = 0$, or $u_z \neq 0$ and $u_x = u_y = 0$, in both cases the length of the rod remains the same. On the other hand if $u_y \neq 0$ and $u_z = u_x = 0$ we will have a length contraction formula equivalent to the one in Eq. (23) where u_x is replaced by u_y . The $HOM(2)$ transformations have a preferred axis which is along the z -axis and consequently the length transformation formulas along these two directions are similar.

Lastly we come to the case where the rod at rest in the S frame is along the z axis with coordinates of its ends given by (x_1, y_1, z_1) and (x_1, y_2, z_2) . In this case if the length of the rod at rest is given by $l_0 = z_2 - z_1$ then, using Eq.(17), Eq. (18) and Eq. (19), it can be shown that

$$l = l_0 \sqrt{\frac{u_x^2(1 - \mathbf{u}^2)}{(1 + u_z)^2} + \frac{u_y^2(1 - \mathbf{u}^2)}{(1 + u_z)^2} + (1 - \mathbf{u}^2)}, \quad (25)$$

where l is as given in Eq. (22). It is immediately observed that the length transformation formula for the third case is different from the previous two cases. This has to do with the special status of the z -axis in $HOM(2)$ and in general in VSR. If $u_x \neq 0$ and $u_y = u_z = 0$, or $u_y \neq 0$ and $u_z = u_x = 0$, in both cases the length of the rod remains the same. On the other hand if $u_z \neq 0$ and $u_x = u_y = 0$ we will have a length contraction formula equivalent to the one in SR as

$$l = l_0 \sqrt{1 - u_z^2}. \quad (26)$$

This formula, corresponding to relative motion along z -axis, again shows the special status of the preferred axis. If fractional length contraction is defined by $\Delta l/l_0$ where $\Delta l \equiv l_0 - l$, for the case of VSR and SR then the contents of Eq. (23) and Eq. (26) can be plotted to show the difference of VSR and SR. Such a plot is given in Fig. 1.

It can be checked, using the fact that $|\mathbf{u}|^2 < 1$, that the expressions in Eq. (21), Eq. (24) and Eq. (25) gives length contractions. Although VSR transformations constitute only a small subgroup of the total Lorentz group but here the length of a moving rod never expands.

3.2 The rod is at rest in the S' frame

In this case we suppose that a rod is at rest along the x' -axis in the S' frame. The length of the rod is $\Delta x' = x'_2 - x'_1 \equiv l_0$. An observer in the S frame, which is moving with a

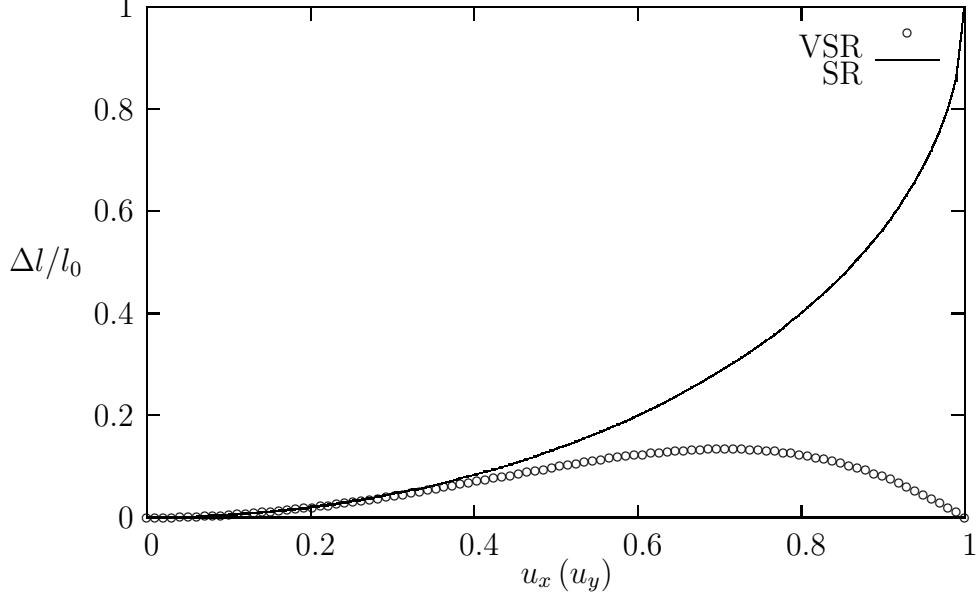


Figure 1: The plot of the $\Delta l/l_0$, where $\Delta l \equiv l_0 - l$, for the case of VSR and SR when the rod is at rest in S frame. In the abscissa we have either u_x (when $u_y = u_z = 0$) or u_y (when $u_x = u_z = 0$).

velocity $-\mathbf{u}$ with respect to the S frame, can measure the length of the rod in his/her frame. For the measurement of the length of the rod in motion one has to know the coordinates of the two ends of the rod $((x_1, y_1, z_1)$ and $(x_2, y_2, z_2))$ simultaneously (at t). From the form of $L(u)$, in Eq. (8), we can write

$$\begin{aligned} x'_1 &= -\gamma_u u_x t + x_1 + \gamma_u u_x z_1, \\ y'_1 &= -\gamma_u u_y t + y_1 + \gamma_u u_y z_1, \\ z'_1 &= -\gamma_u u_z t - \frac{u_x}{1+u_z} x_1 - \frac{u_y}{1+u_z} y_1 + \frac{\gamma_u (1 - \mathbf{u}^2 + u_z + u_z^2)}{(1+u_z)} z_1, \end{aligned}$$

which connects the spatial coordinates of one end of the rod in the two frames. For the the other end we must have

$$\begin{aligned} x'_2 &= -\gamma_u u_x t + x_2 + \gamma_u u_x z_2, \\ y'_2 &= -\gamma_u u_y t + y_2 + \gamma_u u_y z_2, \\ z'_2 &= -\gamma_u u_z t - \frac{u_x}{1+u_z} x_2 - \frac{u_y}{1+u_z} y_2 + \frac{\gamma_u (1 - \mathbf{u}^2 + u_z + u_z^2)}{(1+u_z)} z_2. \end{aligned}$$

Subtracting the first triplet of equations from the second triplet we have

$$l_0 = \Delta x + \gamma_u u_x \Delta z, \quad (27)$$

$$0 = \Delta y + \gamma_u u_y \Delta z, \quad (28)$$

$$0 = -\frac{u_x}{1+u_z} \Delta x - \frac{u_y}{1+u_z} \Delta y + \frac{\gamma_u (1 - \mathbf{u}^2 + u_z + u_z^2)}{(1+u_z)} \Delta z, \quad (29)$$

where $\Delta x \equiv x_2 - x_1$, $\Delta y \equiv y_2 - y_1$ and $\Delta z \equiv z_2 - z_1$. From the last set of equations one obtains

$$\Delta z = \frac{l_0 u_x}{\gamma_u (1 + u_z)}. \quad (30)$$

If the length of the moving rod be

$$l \equiv \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}, \quad (31)$$

then in this case we will have

$$l = l_0 \sqrt{\left(1 - \frac{u_x^2}{1 + u_z}\right)^2 + \frac{u_x^2 u_y^2}{(1 + u_z)^2} + u_x^2 \frac{(1 - \mathbf{u}^2)}{(1 + u_z)^2}}. \quad (32)$$

The equation is not equivalent to Eq. (21) showing that length contraction depends upon the frame in VSR. If the rod was kept along the y' direction in the S' frame, then $l_0 = y'_2 - y'_1$, we would expect the length contraction formula to be exactly same as that given above except an interchange in u_x and u_y . If $u_x = 0$ then there is no length transformation. If on the other hand $u_x \neq 0$ but $u_y = u_z = 0$ then the above formula becomes

$$l = l_0 \sqrt{1 - u_x^2}, \quad (33)$$

which matches exactly with the corresponding result from SR. If on the other hand the rod is placed along the y' axis and $u_y \neq 0$ but $u_z = u_x = 0$ then also we get a length contraction formula exactly similar to Eq. (33) except that there u_x is replaced by u_y . In both these cases it is observed that we get the same results as we get from SR.

If the rod in the S' frame is placed along the z axis it can be easily found out that in this case

$$l = l_0 \sqrt{u_x^2 + u_y^2 + (1 - \mathbf{u})^2}, \quad (34)$$

where l is as given in Eq. (31) and $l_0 = z'_2 - z'_1$. If in this case if $u_z \neq 0$ but $u_x = u_y = 0$ we again get back the SR formula as $l = l_0 \sqrt{1 - u_z^2}$. In this case also one can check, using the fact that $|\mathbf{u}|^2 < 1$, that Eq. (32) and Eq. (34) gives length contractions.

4 Discussion and conclusion

From the above analysis of length transformations in VSR we see that in general the VSR results and SR result do not match. But there are remarkable similarities which may hinder one from discovering the difference in the results predicted by the two theories. In our convention there are two frames S and S' which coincides with each other at $t = t' = 0$. As time evolves S' frame moves relative to S frame with an uniform 3-velocity \mathbf{u} . If the rod is kept along any axes in the S' frame and one measures its length from S frame then the length transformations do not coincide with the SR results. But interestingly if the rod is sliding along any common axes of S and S' frame then we get the exact SR length contraction results. Consequently if the observer is in S frame and the rod is oriented along any 3-axes, where the particular axis is along the direction of the relative velocity \mathbf{u} , one will never discover whether the theory of relativity is SR or VSR. This difference between SR and VSR becomes more blurred because as in SR in VSR also the velocity of light is independent of the reference frame and the time-dilation formula is exactly the same as in SR.

On the other hand if the rod is at rest in the S frame itself and the observer is in the S' frame then the length transformation formulas are different from SR if the orientation

of the rod and the relative velocity is along the x or y axes. But if the rod is placed along the z axis in the S frame and the frame S' also moves along the z axis of the S frame with velocity u_z then the length of the rod measured in the S' frame is contracted in the same way as one expects from SR.

From the forms of $L(u)$ and $L^{-1}(u)$ as given in Eq. (8) and Eq. (11) it is observed that $L^{-1}(u)$ is not obtained from $L(u)$ by putting a minus sign in front of all the velocity components appearing in the expression of $L(u)$. This is the cause of different length contraction formulas for two different frames as shown in the last section. The interesting thing about VSR is that inspite of these *asymmetric* nature of the transformations, the square of the line element remains invariant as in SR and consequently the time-dilation formulas remain exactly the same as in SR. The length contraction formulas in VSR do depend upon the sign of u_z which shows that the amount of contraction of length of a moving rod depends upon its direction of motion along the z or z' -axes.

If VSR is really the theory which nature follows, may be in the very high energy sector or near the electro-weak symmetry breaking scale, then one may hope to see the effects of VSR length contractions in the LHC or in future colliders. At present there is no confirmation of any difference of the length contraction results obtained from SR. In nearby future heavy ion collision experiments and other related experiments can really be done to look for any discrepancy of the length contraction formulas from SR.

In conclusion it must be stated that in this article we have studied how a moving rod's length changes from its rest length in VSR and specifically in the $HOM(2)$ version of VSR. Length contraction is observed for all the cases but there are some variation in the transformation equations in contrast to that in SR, although SR results are reproduced in many special cases of VSR. The other important conclusion is related to the fact that in general the phenomenon of length contraction of a rod in VSR do depend upon the frame from which one observes, a fact which is very difficult to accept in any relativistic theory.

Appendix

A $HOM(2)$ transformations

The VSR generators are given by $T_1 = K_x + J_y$, $T_2 = K_y - J_x$ and K_z where K_i 's and J_i 's are the generators of Lorentz boosts and 3-space rotations of the full Lorentz group. In this article we choose the following form of the generators of \mathbf{J} and \mathbf{K} as given in the book by L. H. Ryder [6]:

$$J_x = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_y = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_z = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$K_x = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently one must have

$$T_1 = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T_2 = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Noting that

$$T_1^2 = - \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad T_2^2 = - \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$

and $T_1^3 = T_2^3 = 0$ we have

$$e^{i\alpha T_1} = \begin{pmatrix} 1 + \frac{\alpha^2}{2!} & \alpha & 0 & -\frac{\alpha^2}{2!} \\ \alpha & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \\ \frac{\alpha^2}{2!} & \alpha & 0 & 1 - \frac{\alpha^2}{2!} \end{pmatrix}, \quad e^{i\beta T_2} = \begin{pmatrix} 1 + \frac{\beta^2}{2!} & 0 & \beta & -\frac{\beta^2}{2!} \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & -\beta \\ \frac{\beta^2}{2!} & 0 & \beta & 1 - \frac{\beta^2}{2!} \end{pmatrix}. \quad (35)$$

The square of K_z is given by

$$K_z^2 = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and it comes out trivially that $K_z^3 = -K_z$. From these facts one can write

$$e^{i\phi K_z} = \begin{pmatrix} \cosh \phi & 0 & 0 & \sinh \phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \phi & 0 & 0 & \cosh \phi \end{pmatrix}. \quad (36)$$

Using the above forms of the matrices we can now calculate $L(u) = e^{i\alpha T_1} e^{i\beta T_2} e^{i\phi K_z}$ which comes out as

$$L(u) = \begin{pmatrix} \cosh \phi + \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi} & \alpha & \beta & \sinh \phi - \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi} \\ \alpha e^{-\phi} & 1 & 0 & -\alpha e^{-\phi} \\ \beta e^{-\phi} & 0 & 1 & -\beta e^{-\phi} \\ \sinh \phi + \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi} & \alpha & \beta & \cosh \phi - \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi} \end{pmatrix}, \quad (37)$$

where we have used the relation

$$e^{-\phi} = \cosh \phi - \sinh \phi.$$

As $L(u)u_0 = u$ where u_0 and u are as given in Eq. (2), we get the following set of equations

$$\gamma_u = \cosh \phi + \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi}, \quad (38)$$

$$\gamma_u u_x = -\alpha e^{-\phi}, \quad (39)$$

$$\gamma_u u_y = -\beta e^{-\phi}, \quad (40)$$

$$\gamma_u u_z = -\sinh \phi - \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi}. \quad (41)$$

Adding Eq. (41) and Eq. (38) we get

$$e^{-\phi} = \gamma_u(1 + u_z).$$

Taking logarithm of both sides we get

$$\phi = -\ln[\gamma_u(1 + u_z)]. \quad (42)$$

Using the expression of $e^{-\phi}$ in Eq. (39) and Eq. (40) we get

$$\alpha = -\frac{u_x}{1 + u_z}, \quad \beta = -\frac{u_y}{1 + u_z}. \quad (43)$$

Remembering that $L(u)$ is a 4 matrix its individual matrix elements can be written as $L_{n,m}$. As in SR, we want to specify all the matrix elements $L_{n,m}$ in terms of the velocity components u_i where $i = x, y, z$. There remains two matrix elements of $L(u)$, $L_{1,4}$ and $L_{4,4}$, in Eq. (37) which requires some dressing up before they can be expressed in terms of the velocity components. Adding Eq. (38) to $L_{1,4} = \sinh \phi - \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi}$ we get $L_{1,4} + \gamma_u = e^{\phi}$. Some trivial manipulation then yields

$$L_{1,4} = -\frac{\gamma_u(u^2 + u_z)}{1 + u_z}. \quad (44)$$

In a similar fashion subtracting Eq. (41) from the expression of $L_{4,4} = \cosh \phi - \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi}$ we get

$$L_{4,4} = \frac{\gamma_u(1 - u^2 + u_z - u_z^2)}{1 + u_z}. \quad (45)$$

Ultimately using Eq. (38) to Eq. (41) and Eq. (44) and Eq. (45) one can write $L(u)$ purely in terms of the velocity components as written in Eq. (8).

B $HOM(2)$ inverse transformations

The $HOM(2)$ inverse transformation is

$$L^{-1}(u) = e^{-i\phi K_z} e^{-i\beta T_2} e^{-i\alpha T_1}.$$

The form of the matrices $e^{-i\phi K_z}$, $e^{-i\beta T_2}$ and $e^{-i\alpha T_1}$ can be obtained from Eq. (35) and Eq. (36) by the following replacements: $\alpha \rightarrow -\alpha$, $\beta \rightarrow -\beta$ and $\phi \rightarrow -\phi$. Multiplying $e^{-i\phi K_z}$, $e^{-i\beta T_2}$ and $e^{-i\alpha T_1}$ in the specific order, as required for the inverse transformation, we get

$$L^{-1}(u) = \begin{pmatrix} \cosh \phi + \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi} & -\alpha e^{-\phi} & -\beta e^{-\phi} & -\sinh \phi - \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi} \\ -\alpha e^{-\phi} & 1 & 0 & \alpha \\ -\beta & 0 & 1 & \beta \\ -\sinh \phi + \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi} & -\alpha e^{-\phi} & -\beta e^{-\phi} & \cosh \phi - \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2}\right) e^{-\phi} \end{pmatrix}. \quad (46)$$

Again using Eq. (38) to Eq. (41) and Eq. (44) and Eq. (45) one can write $L^{-1}(u)$ purely in terms of the velocity components as written in Eq. (11). It can easily be checked that the resulting inverse $HOM(2)$ transformations satisfy

$$L^{-1}(u)u = u_0,$$

where u_0 and u are as given in Eq. (2).

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